Stable Models and U_p Slope Calculations

Ken McMurdy
Ramapo College of New Jersey

October 2, 2011

Overview of Talk

Overview of Talk

Part I - Slopes of U_7 Acting on Modular Forms for $\Gamma_1(49)$

(1) Recall Basic Definitions
(2) State Theorem of Kilford-McMurdy
(3) Explicit Example
(4) Proof Sketch

Overview of Talk

Part I - Slopes of U_7 Acting on Modular Forms for $\Gamma_1(49)$

(1) Recall Basic Definitions
(2) State Theorem of Kilford-McMurdy
(3) Explicit Example
(4) Proof Sketch

Part II - Optimal Models for $X_0(p^n)$ for Slope Calculations

(1) Wish List
(2) A Potentially Useful Family
(3) Some properties and an Example

Basic Definitions

$M_k(\Gamma_1(N)), S_k(\Gamma_1(N))$: classical modular forms and cuspforms
$M_k(\Gamma_1(N), \epsilon), S_k(\Gamma_1(N), \epsilon)$: subspaces with specified character

Basic Definitions

$M_k(\Gamma_1(N)), S_k(\Gamma_1(N))$: classical modular forms and cuspforms
$M_k(\Gamma_1(N), \epsilon), S_k(\Gamma_1(N), \epsilon)$: subspaces with specified character

When a prime p divides N, recall that the Hecke operator, U_p, acts on $M_k(\Gamma_1(N))$, preserving these subspaces. The action of U_p on q-expansions at infinity is given by

$$U_p\left(\sum a_n q^n\right) = \sum a_{np} q^n.$$

Basic Definitions

$M_k(\Gamma_1(N)), S_k(\Gamma_1(N))$: classical modular forms and cuspforms
$M_k(\Gamma_1(N), \epsilon), S_k(\Gamma_1(N), \epsilon)$: subspaces with specified character

When a prime p divides N, recall that the Hecke operator, U_p, acts on $M_k(\Gamma_1(N))$, preserving these subspaces. The action of U_p on q-expansions at infinity is given by

$$U_p\left(\sum a_n q^n\right) = \sum a_{np} q^n.$$

Now, let f be a normalized eigenform defined over a number field K, so that a_p is its U_p eigenvalue. Embed K into $\mathbb{C}_p$. Then the **slope** of f is the p-adic valuation of a_p where $v(p) = 1$.

Basic Definitions

$M_k(\Gamma_1(N)), S_k(\Gamma_1(N))$: classical modular forms and cuspforms
$M_k(\Gamma_1(N), \epsilon), S_k(\Gamma_1(N), \epsilon)$: subspaces with specified character

When a prime p divides N, recall that the Hecke operator, U_p, acts on $M_k(\Gamma_1(N))$, preserving these subspaces. The action of U_p on q-expansions at infinity is given by

$$U_p\left(\sum a_n q^n\right) = \sum a_{np} q^n.$$

Now, let f be a normalized eigenform defined over a number field K, so that a_p is its U_p eigenvalue. Embed K into $\mathbb{C}_p$. Then the **slope** of f is the p-adic valuation of a_p where $v(p) = 1$.

Note: The slope depends on both f and the embedding into $\mathbb{C}_p$.

Basic Definitions

$M_k(\Gamma_1(N)), S_k(\Gamma_1(N))$: classical modular forms and cuspforms
$M_k(\Gamma_1(N), \epsilon), S_k(\Gamma_1(N), \epsilon)$: subspaces with specified character

When a prime p divides N, recall that the Hecke operator, U_p, acts on $M_k(\Gamma_1(N))$, preserving these subspaces. The action of U_p on q-expansions at infinity is given by

$$U_p\left(\sum a_n q^n\right) = \sum a_{np} q^n.$$

Now, let f be a normalized eigenform defined over a number field K, so that a_p is its U_p eigenvalue. Embed K into $\mathbb{C}_p$. Then the **slope** of f is the p-adic valuation of a_p where $v(p) = 1$.

Note: The slope depends on both f and the embedding into $\mathbb{C}_p$.

Open Problem: Determine the slopes of $M_k(\Gamma_1(N), \epsilon)$, as a function of (p, k, N, ϵ) and the embedding.

Kilford-McMurdy for $\Gamma_1(49)$

Fix a primitive 42^{nd} root of unity, ζ, and let χ be the Dirichlet character of conductor 49 defined by $\chi(3) = \zeta$. Let K_1 and K_2 be the 7-adic completions of $\mathbb{Q}[\zeta]$ so that $v(\zeta + 1) > 0$ and $v(\zeta + 4) > 0$ respectively.

Kilford-McMurdy for $\Gamma_1(49)$

Fix a primitive 42^{nd} root of unity, ζ, and let χ be the Dirichlet character of conductor 49 defined by $\chi(3) = \zeta$. Let K_1 and K_2 be the 7-adic completions of $\mathbb{Q}[\zeta]$ so that $v(\zeta + 1) > 0$ and $v(\zeta + 4) > 0$ respectively.

(1) $S_k(\Gamma_1(49), \chi^{7k-6})$ is diagonalized by U_7 over K_1. The slopes of U_7 on this space are the values less than $k - 1$ in

$$\left\{ \frac{1}{6} \cdot \left\lfloor \frac{9i}{7} \right\rfloor : i \in \mathbb{N} \right\}.$$

(2) $S_k(\Gamma_1(49), \chi^{8-7k})$ is diagonalized by U_7 over K_2. The slopes of U_7 on this space are the values less than $k - 1$ in

$$\left\{ \frac{1}{6} \cdot \left\lfloor \frac{9i+6}{7} \right\rfloor : i \in \mathbb{N} \right\}.$$

(Each slope corresponds to a one dimensional eigenspace.)

Example

Let $\psi(3) = \gamma$ a primitive 21^{st} root of unity. Then $S_2(\Gamma_1(49), \psi)$ has one family defined over $\mathbb{Q}(\gamma, \alpha)$ where α is a root of

$$\begin{aligned} x^4 + (\gamma^5 + 1)x^3 + (\gamma^{10} - 5\gamma^5 + 1)x^2 \\ + (\gamma^{11} - 4\gamma^{10} - \gamma^7 - \gamma^6 - 2\gamma^5 - \gamma^3 + 2\gamma^2 - \gamma)x \\ + (2\gamma^{10} + \gamma^9 + \gamma^8 + \gamma^7 - \gamma^6 - \gamma^5 - \gamma^4 + \gamma^2 + \gamma + 1). \end{aligned}$$

Example

Let $\psi(3) = \gamma$ a primitive 21st root of unity. Then $S_2(\Gamma_1(49), \psi)$ has one family defined over $\mathbb{Q}(\gamma, \alpha)$ where α is a root of

$$x^4 + (\gamma^5 + 1)x^3 + (\gamma^{10} - 5\gamma^5 + 1)x^2 + (\gamma^{11} - 4\gamma^{10} - \gamma^7 - \gamma^6 - 2\gamma^5 - \gamma^3 + 2\gamma^2 - \gamma)x + (2\gamma^{10} + \gamma^9 + \gamma^8 + \gamma^7 - \gamma^6 - \gamma^5 - \gamma^4 + \gamma^2 + \gamma + 1).$$

$$a_7 = (\gamma^{11} - \gamma^{10} + \gamma^8 - \gamma^7 - \gamma^6 + \gamma^5 - \gamma^3 + \gamma^2 - 1)\alpha^3 + (\gamma^8 - \gamma^6 + \gamma^5 - \gamma^4 - \gamma^3 + \gamma^2)\alpha^2 + (4\gamma^{11} - \gamma^6 + \gamma^5 + 4\gamma^4 - \gamma^3 + \gamma^2 - \gamma)\alpha - (\gamma^{11} - \gamma^{10} - 3\gamma^9 + \gamma^8 - \gamma^7 - 2\gamma^6 + 2\gamma^5 + \gamma^4 - 3\gamma^3 + 2\gamma^2 + \gamma - 3).$$

Example

Let $\psi(3) = \gamma$ a primitive 21^{st} root of unity. Then $S_2(\Gamma_1(49), \psi)$ has one family defined over $\mathbb{Q}(\gamma, \alpha)$ where α is a root of

$$x^4 + (\gamma^5 + 1)x^3 + (\gamma^{10} - 5\gamma^5 + 1)x^2 + (\gamma^{11} - 4\gamma^{10} - \gamma^7 - \gamma^6 - 2\gamma^5 - \gamma^3 + 2\gamma^2 - \gamma)x + (2\gamma^{10} + \gamma^9 + \gamma^8 + \gamma^7 - \gamma^6 - \gamma^5 - \gamma^4 + \gamma^2 + \gamma + 1).$$

$$a_7 = (\gamma^{11} - \gamma^{10} + \gamma^8 - \gamma^7 - \gamma^6 + \gamma^5 - \gamma^3 + \gamma^2 - 1)\alpha^3 + (\gamma^8 - \gamma^6 + \gamma^5 - \gamma^4 - \gamma^3 + \gamma^2)\alpha^2 + (4\gamma^{11} - \gamma^6 + \gamma^5 + 4\gamma^4 - \gamma^3 + \gamma^2 - \gamma)\alpha - (\gamma^{11} - \gamma^{10} - 3\gamma^9 + \gamma^8 - \gamma^7 - 2\gamma^6 + 2\gamma^5 + \gamma^4 - 3\gamma^3 + 2\gamma^2 + \gamma - 3).$$

The theorem applies over K_1 if we take $\gamma = \zeta^8$, since

$$\chi^{7(2)-6} = \chi^8 = \gamma.$$

Example (cont)

Choose the uniformizer $\pi_1 = -\zeta^8 + \zeta^6 - \zeta^4 + \zeta$ for K_1. Then $v(\pi_1) = 1/6$. The roots for α are defined over K_1 with the following approximations:

$$\begin{aligned}
\alpha_1 &= 4 + 5\pi_1 + 1\pi_1^2 + 2\pi_1^3 + 3\pi_1^4 + 5\pi_1^5 + \cdots \\
\alpha_2 &= 5 + 4\pi_1 + 2\pi_1^2 + 3\pi_1^3 + 4\pi_1^4 + 1\pi_1^5 + \cdots \\
\alpha_3 &= 4 + 1\pi_1 + 5\pi_1^2 + 4\pi_1^3 + 1\pi_1^4 + 6\pi_1^5 + \cdots \\
\alpha_4 &= 5 + 5\pi_1^2 + 4\pi_1^3 + 4\pi_1^5 + 2\pi_1^6 + \cdots
\end{aligned}$$

Plugging these values into a_7 we find π_1-adic valuations of 1, 2, 3, and 5. So the theorem is verified in this special case.

Outline of the Proof

In order to use p-adic analysis to prove results about slopes of classical modular eigenforms for $\Gamma_1(N)$, these are the standard steps:

Outline of the Proof

In order to use p-adic analysis to prove results about slopes of classical modular eigenforms for $\Gamma_1(N)$, these are the standard steps:

(1) Use the geometry of $X_1(N)$ to embed $M_k(\Gamma_1(N))$ into a natural p-adic family of modular forms ("overconvergent" modular forms). Verify that U_p extends to the family.

Outline of the Proof

In order to use p-adic analysis to prove results about slopes of classical modular eigenforms for $\Gamma_1(N)$, these are the standard steps:

(1) Use the geometry of $X_1(N)$ to embed $M_k(\Gamma_1(N))$ into a natural p-adic family of modular forms ("overconvergent" modular forms). Verify that U_p extends to the family.

(2) Compute the U_p slopes of all of the overconvergent eigenforms in your family using analytic techniques.

Outline of the Proof

In order to use p-adic analysis to prove results about slopes of classical modular eigenforms for $\Gamma_1(N)$, these are the standard steps:

(1) Use the geometry of $X_1(N)$ to embed $M_k(\Gamma_1(N))$ into a natural p-adic family of modular forms ("overconvergent" modular forms). Verify that U_p extends to the family.

(2) Compute the U_p slopes of all of the overconvergent eigenforms in your family using analytic techniques.

(3a) Use a theorem of Coleman to conclude that all of the overconvergent eigenforms of small slope are classical. Thus, you have constructed a certain number of classical eigenforms with specified slopes.

Outline of the Proof

In order to use p-adic analysis to prove results about slopes of classical modular eigenforms for $\Gamma_1(N)$, these are the standard steps:

(1) Use the geometry of $X_1(N)$ to embed $M_k(\Gamma_1(N))$ into a natural p-adic family of modular forms ("overconvergent" modular forms). Verify that U_p extends to the family.

(2) Compute the U_p slopes of all of the overconvergent eigenforms in your family using analytic techniques.

(3a) Use a theorem of Coleman to conclude that all of the overconvergent eigenforms of small slope are classical. Thus, you have constructed a certain number of classical eigenforms with specified slopes.

(3b) Compare with known formulas for the total number of classical eigenforms with a given character.

Outline of the Proof

In order to use p-adic analysis to prove results about slopes of classical modular eigenforms for $\Gamma_1(N)$, these are the standard steps:

(1) Use the geometry of $X_1(N)$ to embed $M_k(\Gamma_1(N))$ into a natural p-adic family of modular forms ("overconvergent" modular forms). Verify that U_p extends to the family.

(2) Compute the U_p slopes of all of the overconvergent eigenforms in your family using analytic techniques.

(3a) Use a theorem of Coleman to conclude that all of the overconvergent eigenforms of small slope are classical. Thus, you have constructed a certain number of classical eigenforms with specified slopes.

(3b) Compare with known formulas for the total number of classical eigenforms with a given character.

(4) Keep fingers crossed that (3a) and (3b) are the same!!

Overconvergent Modular Forms of Level 49

Our overconvergent forms are defined *intrinsically*, following Coleman's construction in "Classical and Overconvergent Modular Forms of Higher Level."

Overconvergent Modular Forms of Level 49

Our overconvergent forms are defined *intrinsically*, following Coleman's construction in "Classical and Overconvergent Modular Forms of Higher Level."

Let $f : E_1(49) \to X_1(49)$ be the universal generalized elliptic curve over $X_1(49)$, and $\omega = f_* \Omega^1_{E_1(49)/X_1(49)}$. Then $M_k(\Gamma_1(49))$ is just the holomorphic sections of $\omega^{\otimes^k}$. Overconvergent forms are holomorphic sections over $W = W_1(49)$, a certain wide-open neighborhood of the cusp ∞, in the analytification of the curve.

Overconvergent Modular Forms of Level 49

Our overconvergent forms are defined *intrinsically*, following Coleman's construction in "Classical and Overconvergent Modular Forms of Higher Level."

Let $f : E_1(49) \to X_1(49)$ be the universal generalized elliptic curve over $X_1(49)$, and $\omega = f_*\Omega^1_{E_1(49)/X_1(49)}$. Then $M_k(\Gamma_1(49))$ is just the holomorphic sections of $\omega^{\otimes^k}$. Overconvergent forms are holomorphic sections over $W = W_1(49)$, a certain wide-open neighborhood of the cusp ∞, in the analytification of the curve.

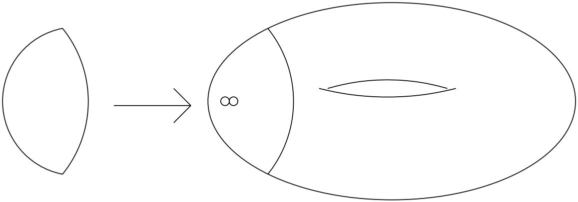
We want to work on $X_0(49)$, because we have good explicit equations. Fortunately, there are Eisenstein series, $E_{1,\chi}$ and $E_{1,\tau}$ which are holomorphic and non-vanishing over W. Therefore, we can define an isomorphism

$$M_0(\Gamma_0(49))(D) \cong M_k(\Gamma_1(49), \chi\tau^{k-1})(W),$$

where D is the wide open disk of $X_0(49)$ over which W lies (via the forgetful map).

General Setup - The Picture

$$\begin{array}{ccc} W & \longrightarrow & X_1(49) \\ \downarrow & & \downarrow \\ D & \longrightarrow & X_0(49) \end{array}$$



$$M_0(\Gamma_0(49))(D) \to M_k(\Gamma_1(49), \chi\tau^{k-1})(W)$$

$$f \mapsto f \cdot E_{1,\chi} \cdot E_{1,\tau}^{k-1}$$

Let $\widetilde{U}_7$ be the induced linear operator on $M_0(\Gamma_0(49))(D)$.

The Explicit Part of the Proof

Now we consider the following explicit model for $X_0(49)$.

$$\begin{aligned} y^2 - 7xy(x^2+5x+7) \\ - x(x^6+7x^5+21x^4+49x^3+147x^2+343x+343) = 0 \\ z^2 = x(4x^2+21x+28) \end{aligned}$$

Here, $x = \eta_1/\eta_{49}$ and $y = \eta_7^4/\eta_{49}^4$. Also, $t = x^4/y$ is a parameter on the genus 0 curve, $X_0(7)$, with divisor $(0) - (\infty)$, which lifts to a parameter on D.

The Explicit Part of the Proof

Now we consider the following explicit model for $X_0(49)$.

$$\begin{aligned} y^2 - 7xy(x^2+5x+7) \\ - x(x^6+7x^5+21x^4+49x^3+147x^2+343x+343) &= 0 \\ z^2 &= x(4x^2+21x+28) \end{aligned}$$

Here, $x = \eta_1/\eta_{49}$ and $y = \eta_7^4/\eta_{49}^4$. Also, $t = x^4/y$ is a parameter on the genus 0 curve, $X_0(7)$, with divisor $(0) - (\infty)$, which lifts to a parameter on D.

Taking $s = \sqrt[4]{7}/t$, $M_0(\Gamma_0(49))(D)$ has "basis" $\{s, s^2, s^3, \cdots\}$. Every form has a unique power series expansion in s, and the forms of bounded norm are given by $\mathbb{R}_7[[s]] \otimes \mathbb{C}_7$. This implies that the characteristic polynomials of the truncations of the corresponding matrix representing $\tilde{U}_7$ converge in sup norm to the characteristic series of $\tilde{U}_7$.

A Truncation of the Large Matrix ($k = 1$ shown)

Write $\tilde{U}_7(s^i)$ as a power series in s, and put the coefficients in the i^{th} column. This yields an infinite dimensional matrix that represents $\tilde{U}_7$ in the basis $\{s, s^2, \ldots\}$. A truncation of the corresponding matrix of 7-adic valuations, over K_1, is as follows.

$$\begin{bmatrix} 1/6 & 5/12 & 1/2 & 3/4 & 1 & 5/4 & 3/2 \\ 1/4 & 1/3 & 7/12 & 5/6 & 13/12 & 7/6 & 29/12 \\ 1/6 & 5/12 & 2/3 & 11/12 & 1 & 5/4 & 5/2 \\ 1/4 & 1/2 & 3/4 & 5/6 & 13/12 & 4/3 & 31/12 \\ 1/6 & 5/12 & 1/2 & 3/4 & 1 & 5/4 & 7/3 \\ 1/4 & 1/3 & 7/12 & 5/6 & 13/12 & 7/6 & 29/12 \\ 1/6 & 5/12 & 2/3 & 11/12 & 1 & 5/4 & 5/2 \end{bmatrix}$$

Our theorem says that the sequence of slopes should be $\{1/6, 1/3, 1/2, 5/6, 1, 7/6, 3/2, \ldots\}$ (almost the sequence of column valuations). This will follow if the determinant of each $j \times j$ truncation is larger than that of any other principle $j \times j$ minor. To prove that, we consider the “column functions.”

"Column Functions"

Proposition: Approximations for $\widetilde{U}_7(s^i)$ for $1 \leq i \leq 7$ over $K_1(\alpha)$ where $\alpha^4 = -7$ are as follows.

$$\begin{aligned}
\widetilde{U}_7(s^1) &\equiv 2\alpha\pi_1 z/(x(x+\pi_1^3)), & \mathbf{v}_1 &= 2, & \mathbf{e}_1 &\geq 3\\
\widetilde{U}_7(s^2) &\equiv 4\alpha^2\pi_1^2/x, & \mathbf{v}_1 &= 4, & \mathbf{e}_1 &\geq 5\\
\widetilde{U}_7(s^3) &\equiv \alpha^3 z/x^2 + 5\alpha^3\pi_1^2/x, & \mathbf{v}_1 &= 6, & \mathbf{e}_1 &\geq 8\\
\widetilde{U}_7(s^4) &\equiv 3\alpha^4 z/x^2 + 2\alpha^4\pi_1^2(x+4\pi_1^3)/x^2, & \mathbf{v}_1 &= 9, & \mathbf{e}_1 &\geq 11\\
\widetilde{U}_7(s^5) &\equiv 6\alpha^5 z(x+\pi_1^3)/x^3, & \mathbf{v}_1 &= 12, & \mathbf{e}_1 &\geq 13\\
\widetilde{U}_7(s^6) &\equiv \alpha^6\pi_1(x^2+7)/x^3, & \mathbf{v}_1 &= 14, & \mathbf{e}_1 &\geq 15\\
\widetilde{U}_7(s^7) &\equiv \alpha^7/t, & \mathbf{v}_1 &= 18, & \mathbf{e}_1 &\geq 19
\end{aligned}$$

(A recursive formula kicks in from there.)

Note: $\frac{1}{12}\mathbf{v}_1(f)$ denotes the minimal 7-adic valuation of f over D.

“Column Functions (cont)”

Scaling and reducing the column functions on the stable reduction, we have the following functions and divisors.

$$
\begin{aligned}
(Z/(X(X-1))) &= (\infty) + (-1,0) - (0,0) - (1,0) \\
(1/X) &= 2(\infty) - 2(0,0) \\
(Z/X^2) &= (1,0) + (-1,0) + (\infty) - 3(0,0) \\
((X-1)/X^2) &= 2(1,0) + 2(\infty) - 4(0,0) \\
(Z(X-1)/X^3) &= 3(1,0) + (-1,0) + (\infty) - 5(0,0) \\
((X^2-1)/X^3) &= 2(1,0) + 2(-1,0) + 2(\infty) - 6(0,0) \\
(Z(X^2-1)/X^4) &= 3(1,0) + 3(-1,0) + (\infty) - 7(0,0).
\end{aligned}
$$

By Riemann-Roch, no linear combination of the first j can ever vanish to degree $j+1$ at ∞. Thus, the determinant of the j^{th} truncation approximates the j^{th} coefficient of the characteristic series and the slopes are as claimed.

Part II - Optimal Models for $X_0(p^n)$ for Slope Calculations

Optimal Models for $X_0(p^n)$ for Slope Calculations

In order to make a similar slope argument more generally, we would need a model with the following properties.

Optimal Models for $X_0(p^n)$ for Slope Calculations

In order to make a similar slope argument more generally, we would need a model with the following properties.

(1) We must be able to write down a “Banach basis” for the functions on $W_1(p^n)$.

Optimal Models for $X_0(p^n)$ for Slope Calculations

In order to make a similar slope argument more generally, we would need a model with the following properties.

(1) We must be able to write down a "Banach basis" for the functions on $W_1(p^n)$.

Canonical Example: Let W be the wide open in $\mathbf{P}^1$ whose $\mathbb{C}_p$-valued points satisfy

$$v((x-1)(x-2)(x-3)) < 1$$

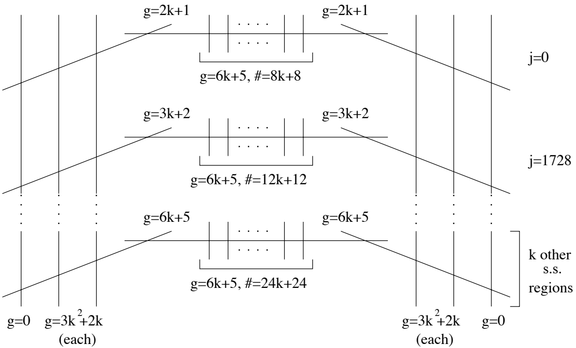
(the complement of three affinoid disks). Then

$$A_K(W) = K < X, Y, Z > /(XY - p(X-Y), 2XZ - p(X-Z), YZ - p(Y-Z)).$$

Think $X = \frac{p}{t-1}$, $Y = \frac{p}{t-2}$ and $Z = \frac{p}{t-3}$ for a parameter t on $\mathbf{P}^1$.

In general, $W_1(p^n)$ is isomorphic to the complement in $\mathbf{P}^1$ of ss affinoid disks (one for each supersingular j-invariant).

(2) Parameters should generate the Weierstrass parameters on the “first” supersingular components.



Stable reduction of $X_0(p^3)$ when $p = 12k + 11$ is shown. The left-most genus 0 vertical component is the reduction of $W_1(p^3)$. It intersects the components, $\mathbf{Y}^A_{2\,1}$, which have the equation

$$y^2 = x^{(p+1)/i(A)} - 1.$$

Candidate Model for $X_0(p)$

$$t = \left(\frac{\eta_1}{\eta_p}\right)^{e_1} \qquad x = \left(\frac{dt/t}{(\eta_1\eta_p)^2}\right)^{e_2}$$

If $p = 12k + 1$, we have: $(e_1, e_2) = (2, 6)$ and

$$(t) = k(0) - k(\infty)$$
$$(x)_{neg} = -(6k+1)(0) - (6k+1)(\infty).$$

If $p = 12k + 5$, we have $(e_1, e_2) = (6, 2)$ and

$$(t) = (3k+1)(0) - (3k+1)(\infty)$$
$$(x)_{neg} = -(2k+1)(0) - (2k+1)(\infty).$$

If $p = 12k + 7$, we have $(e_1, e_2) = (4, 3)$ and

$$(t) = (2k+1)(0) - (2k+1)(\infty)$$
$$(x)_{neg} = -(3k+2)(0) - (3k+2)(\infty).$$

If $p = 12k + 11$, we have $(e_1, e_2) = (12, 1)$ and

$$(t) = (6k+5)(0) - (6k+5)(\infty)$$
$$(x)_{neg} = -(k+1)(0) - (k+1)(\infty).$$

Properties and Example

Important Fact: The Atkin-Lehner involution, w_1, fixes x and satisfies

$$w_1^* t = \frac{p^{(e_1/2)}}{t}.$$

Example: $X_0(17)$ has the equation:

$$\begin{aligned} t^3x^4 + (-3934t^3)x^3 + (-8608t^4 + 2667641t^3 - 42291104t^2)x^2 \\ + (-2944t^5 - 408968t^4 - 38771644t^3 - 2009259784t^2 - 71061003136t)x \\ - 256t^6 - 79328t^5 - 11950529t^4 - 1059834654t^3 \\ - 58712948977t^2 - 1914785073632t - 30358496383232 = 0 \end{aligned}$$

It's actually much nicer. For example, $f(0,t) = t^3 \cdot [g(t) + g(\frac{17^3}{t})]$, where

$$g(t) = -256t^3 - 79328t^2 - 11950529t - 529917327.$$

It's almost certainly possible to compute slopes for specific p using this model - less clear what can be done in general.