The distribution of some arithmetic sequences in arithmetic progressions to large moduli

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Primes

To count primes, we usually define

$$\pi(x) := \#\{p \leq x\}.$$

For technical reasons, we add the weight $\log p$ at each prime p.

Definition

$$\psi(x) := \sum_{p^k \leq x} \log p,$$

$$\psi(x; q, a) := \sum_{\substack{p^k \leq x \\ p^k \equiv a \bmod q}} \log p.$$

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The prime number theorem in arithmetic progressions

Theorem (Hadamard, de la Vallée-Poussin)

If $(a, q) = 1$,

$$\psi(x; q, a) \sim \frac{\psi(x)}{\phi(q)}.$$

This is for fixed values of a and q.

What if we want to look at higher moduli ?

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Higher moduli

Theorem (Siegel, Walfisz)

If $(a, q) = 1$, then for $q \leq (\log x)^B$,

$$\left|\psi(x; q, a) - \frac{\psi(x)}{\phi(q)}\right| \leq C\frac{x}{(\log x)^A}.$$

Theorem

Assume GRH. If $(a, q) = 1$, then

$$\left|\psi(x; q, a) - \frac{\psi(x)}{\phi(q)}\right| \leq C\sqrt{x}(\log x)^2.$$

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The Riemann hypothesis is true on average

On average, we know much more.

Theorem (Bombieri, Vinogradov)

For $Q \leq x^{\frac{1}{2}-\epsilon}$,

$$\sum_{q \leq Q} \max_{a:(a,q)=1} \left| \psi(x;q,a) - \frac{\psi(x)}{\phi(q)} \right| \leq C \frac{x}{(\log x)^A}.$$

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An asymptotic for the mean

Instead of looking at the mean deviation, look at the mean itself.

Theorem (F.)

Let $a \neq 0$, and $M = M(x) \leq (\log x)^B$. We have that

$$\frac{1}{\frac{x}{M}\frac{\phi(a)}{a}} \sum_{\substack{q \leq \frac{x}{M} \\ (q,a)=1}} \left(\psi(x;q,a) - \Lambda(a) - \frac{\psi(x)}{\phi(q)} \right) = \mu(a,M) + O\left(\frac{1}{M^{\frac{205}{538}-\epsilon}} \right)$$

where

$$\mu(a,M) := \begin{cases} -\frac{1}{2}\log M - C_5 & \text{if } a = \pm 1, \\ -\frac{1}{2}\log p & \text{if } a = \pm p^e, \\ 0 & \text{if } a \text{ has } \geq 2 \text{ distinct prime factors.} \end{cases}$$

(The O-constant depends on a, ϵ and B.)

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Notation

Fix a sequence $\mathcal{A} = \{\mathbf{a}(n)\}_{n\geq 1}$ a sequence of non-negative real numbers.

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In each of the sequences we will consider, there exists $g_a(q)$ such that

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Values of a positive definite binary quadratic form, with multiplicity.

For $\alpha, \beta, \gamma \in \mathbb{Z}$ coprime, let

$$Q(x, y) := \alpha x^2 + \beta xy + \gamma y^2$$

be a positive definite binary quadratic form.

The discriminant: $d := \beta^2 - 4\alpha\gamma$.

$$\mathbf{a}(n) := \#\{(x, y) \in \mathbb{Z}^2_{\geq 0} : Q(x, y) = n\}.$$

$r_d(n) := \#$ distinct representations of n by all non-equivalent forms of discriminant d (up to automorphism).

$$\rho_a(q) := \frac{1}{q} \cdot \#\{1 \leq x, y \leq q : Q(x, y) \equiv a \bmod q\}.$$

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What happens in arithmetic progressions ?

$$\mathcal{A}(x;q,a) \sim \frac{\rho_a(q)}{q}\mathcal{A}(x). \tag{1}$$

This asymptotic actually holds in great uniformity.

Theorem (Plaksyn)

The asymptotic (1) holds (with a good error term) for $q \leq x^{2/3}$.

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Assume that $d \equiv 1, 5, 9, 12, 13 \bmod 16$. Fix a such that $(a, 2d) = 1$. We have for $M = M(x) \leq x^{\lambda}$, where $\lambda < \frac{1}{12}$ that

$$\frac{1}{x/M} \sum_{q \leq \frac{x}{M}} \left(\mathcal{A}(x; q, a) - \mathbf{a}(a) - \frac{\rho_a(q)}{q} \mathcal{A}(x) \right)$$

$$= -C_Q \rho_a(4d) r_d(|a|) + O\left(\frac{1}{M^{\frac{1}{3}-\epsilon}} \right),$$

where $C_Q := \frac{A_Q}{2L(1, \chi_d)} \qquad \left(= \frac{w_d \sqrt{|d|}}{4\pi h_d} A_Q \right),$

$A_Q =$ area of $\{(x, y) \in \mathbb{R}^2_{\geq 0} : Q(x, y) \leq 1\}$, $\chi_d := \left(\frac{4d}{\cdot} \right)$, w_d is the number of units of $\mathbb{Q}(\sqrt{d})$ and h_d is its class number.

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Twin primes

$$\mathbf{a}(n) := \Lambda(n)\Lambda(n+2).$$

Conjecture (Hardy-Littlewood)

$$\mathcal{A}(x) \sim 2C_2 x,$$

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The (general) Hardy-Littlewood actually tells something about arithmetic progressions.

For $(a,q)=1$, look at

$$B(x) := \sum_{n\leq x} \Lambda(qn+a)\Lambda(qn+a+2).$$

The Hardy-Littlewood prediction is that

$$B(x) \sim \frac{\mathcal{A}(x)}{\gamma(q)},$$

where

$$\gamma(q) := \prod_{p\mid q}\left(1-\frac{\nu(p)}{p}\right), \qquad \text{where } \nu(p) := \begin{cases} 2 & \text{if } p\neq 2 \\ 1 & \text{if } p=2. \end{cases}$$

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Theorem (F.)

Under a uniform version of Hardy-Littlewood, the average of $\mathcal{A}(x;q,a) - \mathbf{a}(a) - \frac{\mathcal{A}(x)}{q\gamma(q)}$ for $\frac{x}{2M} < q \le \frac{x}{M}$ is

$$\begin{cases} \sim -\frac{(\log M)^2}{4} & \text{if } a = -1 \\ \sim -\frac{\log 3}{4}\log M & \text{if } a = 1, -3 \\ \sim -\frac{\log 2}{2}\log M & \text{if } a = 2, -4 \\ \sim -\frac{\log p \log q}{2}\frac{p-\nu(p)}{p-1}\frac{q-\nu(q)}{q-1} & \text{if } a(a+2) = \pm p^e q^f \\ o(1) & \text{if } \omega(a(a+2)) \ge 3. \end{cases}$$

Of course one can do this with any admissible k-tuple of linear forms in the primes.

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Sums of two squares, without multiplicity

Define

$$\mathbf{a}(n) := \begin{cases} 1 & \text{if } n = \square + \square, \\ 0 & \text{else.} \end{cases}$$

In this case,

$$\mathcal{A}(x; q, a) \sim g_a(q)\mathcal{A}(x),$$

where, for $p \neq 2$ with $p^f \parallel a$,

$$g_a(p^e) := \frac{1}{p^e} \times \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4 \\ 1 & \text{if } p \equiv 3 \bmod 4, e \leq f, 2 \mid e \\ \frac{1}{p} & \text{if } p \equiv 3 \bmod 4, e \leq f, 2 \nmid e \\ 1 + \frac{1}{p} & \text{if } p \equiv 3 \bmod 4, e > f, 2 \mid f \\ 0 & \text{if } p \equiv 3 \bmod 4, e > f, 2 \nmid f. \end{cases} \qquad (2)$$

Moreover, $g_a(2) := \frac{1}{2}$ and for $e \geq 2$, $g_a(2^e) := \frac{1+(-1)^{\frac{a-1}{2}}}{2^{e+2}}$.

Sums of two squares, without multiplicity

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$$\mathbf{a}(n) := \begin{cases} 1 & \text{if } n = \square + \square, \\ 0 & \text{else.} \end{cases}$$

In this case,

$$\mathcal{A}(x; q, a) \sim g_a(q)\mathcal{A}(x),$$

where, for $p \neq 2$ with $p^f \parallel a$,

$$g_a(p^e) := \frac{1}{p^e} \times \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4 \\ 1 & \text{if } p \equiv 3 \bmod 4, e \leq f, 2 \mid e \\ \frac{1}{p} & \text{if } p \equiv 3 \bmod 4, e \leq f, 2 \nmid e \\ 1 + \frac{1}{p} & \text{if } p \equiv 3 \bmod 4, e > f, 2 \mid f \\ 0 & \text{if } p \equiv 3 \bmod 4, e > f, 2 \nmid f. \end{cases} \quad (2)$$

Moreover, $g_a(2) := \frac{1}{2}$ and for $e \geq 2$, $g_a(2^e) := \frac{1+(-1)^{\frac{a-1}{2}}}{2^{e+2}}$.

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Sums of two squares, without multiplicity

Theorem (F.))

Fix an integer $a \equiv 1 \bmod 4$. We have for $1 \le M(x) \le (\log x)^{\lambda}$, where $\lambda < 1/5$ is a fixed real number, that

$$\frac{1}{x/2M} \sum_{\frac{x}{2M} < q \le \frac{x}{M}} \left(\mathcal{A}(x;q,a) - \mathbf{a}(a) - g_a(q)\mathcal{A}(x)\right)$$
$$\sim -\left(\frac{\log M}{\log x}\right)^{\frac{1}{2}} \frac{(-4)^{-l_a-1}(2l_a+2)!}{(4l_a^2-1)(l_a+1)!\pi} \prod_{\substack{p^f \| a:\\ p\equiv 3 \bmod 4,\\ f \text{odd}}} \frac{\log(p^{\frac{f+1}{2}})}{\log M}, \quad (3)$$

where $l_a := \#\{p^f \parallel a : p \equiv 3 \bmod 4, 2 \nmid f\}$ is the number primes dividing a to an odd power which are congruent to 3 modulo 4.

Integers free of small prime factors

For $y = y(x)$ a function of x, define

$$\mathbf{a}_y(n) := \begin{cases} 1 & \text{if } p \mid n \Rightarrow p \geq y \\ 0 & \text{else,} \end{cases}$$

$$\mathcal{A}(x,y) := \sum_{n \leq x} \mathbf{a}_y(n),$$

$$\gamma_y(q) := \prod_{\substack{p \mid q \\ p < y}} \left(1 - \frac{1}{p}\right),$$

$$\mathcal{A}(x,y;q,a) := \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \mathbf{a}_y(n).$$

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Integers free of small prime factors

Theorem (F.)

Let $M \leq (\log x)^{1-\delta}$. The average of $\mathcal{A}(x, y; q, a) - \mathbf{a}_y(a) - \frac{\mathcal{A}(x,y)}{q\gamma_y(q)}$ for $x/2M < q \leq x/M$ is, for $y \leq e^{(\log M)^{\frac{1}{2}-\delta}}$ with $y \to \infty$,

$$= \begin{cases} -\frac{1}{2} + o(1) & \text{if } a = \pm 1 \\ o(1) & \text{otherwise,} \end{cases}$$

and for $(\log x)^{\log\log\log x} \leq y \leq \sqrt{x}$, it is

$$= \frac{\mathcal{A}(x,y)}{x} \times \begin{cases} (-\frac{1}{2} + o(1)) \log M & \text{if } a = \pm 1 \\ -\frac{1}{2} \log p + o(1) & \text{if } a = \pm p^k \\ o(1) & \text{otherwise.} \end{cases}$$

(We have no result in the intermediate range.)

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Fin.