New Percolation Crossing Formulas and Second-order Modular Forms

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This work was supported in part by National Science Foundation Grant No. DMR-0536927

OUTLINE

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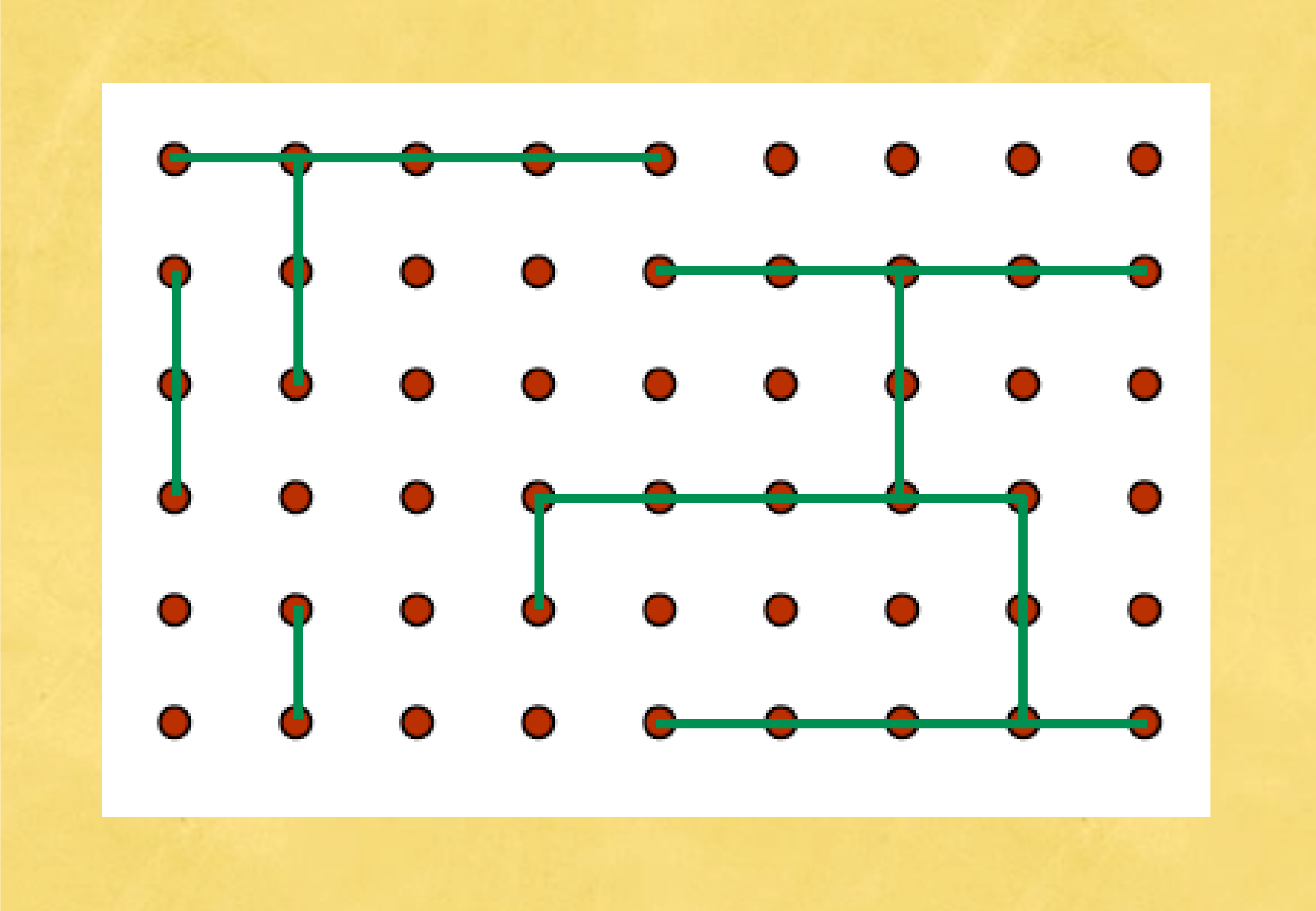
First of all, what IS percolation?

Imagine a large square lattice of points, with (green) bonds — between neighboring points put in place with (independent) probability p. A given configuration might look like this:

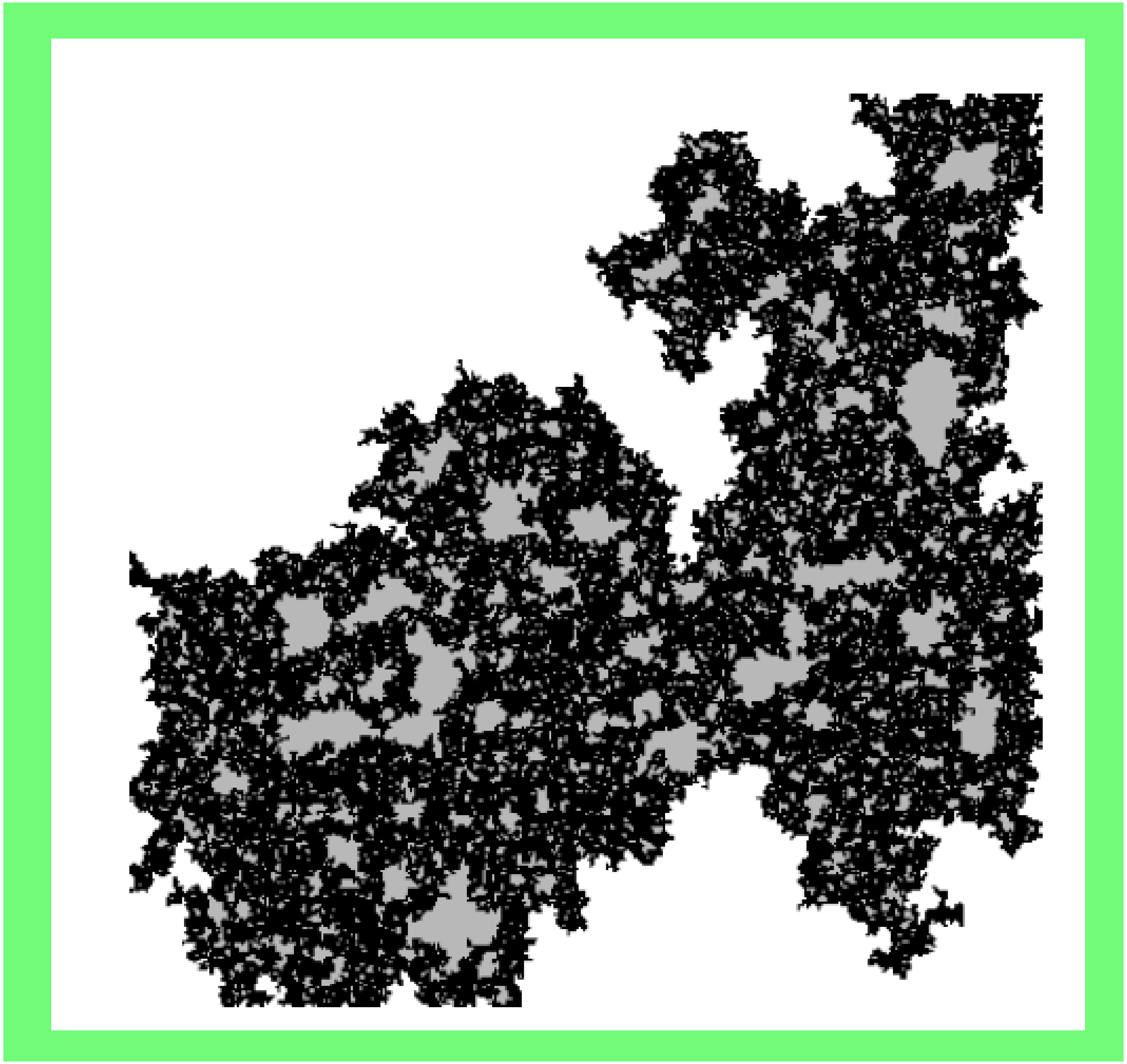


The occupied bonds form clusters. It is the geometric properties of these clusters that are of interest.

When p is small (near 0), the lattice will be mostly empty (for the great majority of configurations). When p is large (near 1), it will be mostly full. If we let the lattice get very large, there is rigorously known to be a phase transition (at $p = p_c = 1/2$ for the bond model shown). The results discussed here are all at p_c.

At p_c, on a large lattice clusters are quite ramified (fractal).

Here is a single cluster:

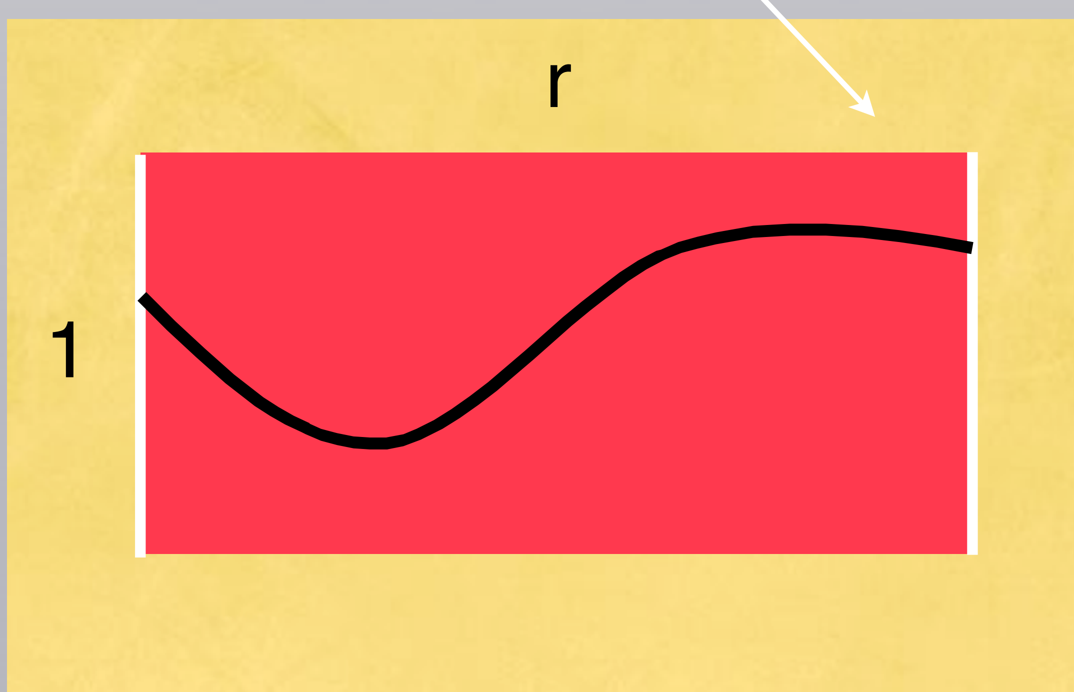


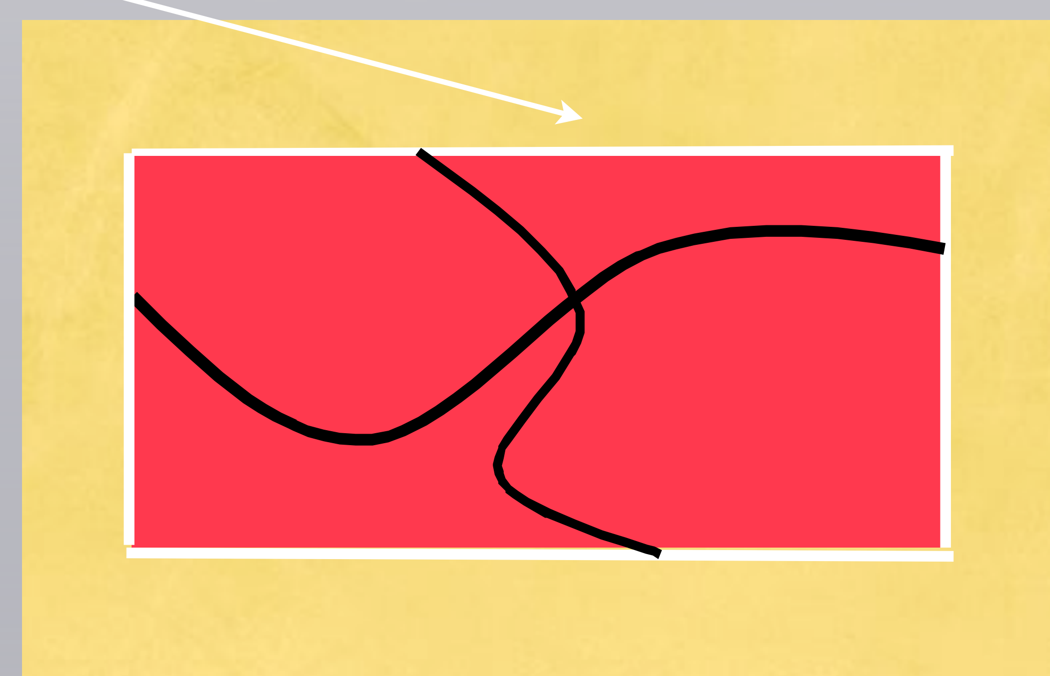
CROSSING PROBABILITIES

For a large, rectangular lattice of aspect ratio r, the crossing probabilities:

$\Pi_h(r)$, the probability of a horizontal crossing (a cluster connecting the two vertical sides), and

$\Pi_{hv}(r)$, the probability of connecting all four sides of the rectangle (horizontal-vertical).





It is convenient to define Π_{hnv}, the probability of a horizontal but not vertical crossing

$$\Pi_{hv} = \Pi_h - \Pi_{hnv}$$

Understanding the phase transition:

For $p < p_c$, clusters are a.s. small, so $\Pi_h(r) = 0$. For $p > p_c$, they are a.s. large, so $\Pi_h(r) = 1$. Only for $p = p_c$ is Π_h a non-trivial function of r.

Explicit formulas for the crossing probabilities were first found by physicists using conformal field theory. Cardy's formula for the rectangle:

$$\Pi_h(r) = \frac{2\pi\sqrt{3}}{\Gamma(1/3)^3}\lambda^{1/3}\ {}_2F_1(1/3, 2/3; 4/3; \lambda).$$

Here the aspect ratio r enters via the cross–ratio λ(r) of the image points in the upper half plane of the four corners under a conformal map.

A rigorous derivation on the triangular lattice was given later by Smirnov, this year's Fields medalist.

Some years ago, Bob Ziff noticed that

$$\Pi'_h(\lambda(r)) = -4\sqrt{3}\,C\,\eta(ir)^4$$

with

$$C := \frac{2^{1/3}\pi^2}{3\,\Gamma(1/3)^3},$$

and η the Dedekind function. Thus, Π'_h is a modular form of weight 2 (on the full modular group).

Modular properties require a function to have certain simple transformation properties under

$S: z \rightarrow -1/z$ (with $z = ir$)

and

$T: z \rightarrow z+1$

(or combinations of these operations). Here, the behavior under S follows directly from the physical symmetries of the problem, but the T behavior comes from the structure of the crossing formulas themselves and has no obvious physical origin.

Explicitly

$$z^{-2}\,\Pi'_h(-1/z) =: \Pi'_h(z)|_2 S = -\Pi'_h(z)$$

$$\Pi'_h(z)|T^2 = e^{2\pi i/3}\,\Pi'_h(z)$$

while

$$\Pi'_{hnv}(z)|_2 S = \Pi'_{hnv}(z) - C\,\Pi'_h(z)$$

$$\Pi'_{hnv}(z)|T^2 = \Pi'_{hnv}(z)$$

The operations S and T^2 generate the theta-group Γ_θ.

(PK and Don Zagier)

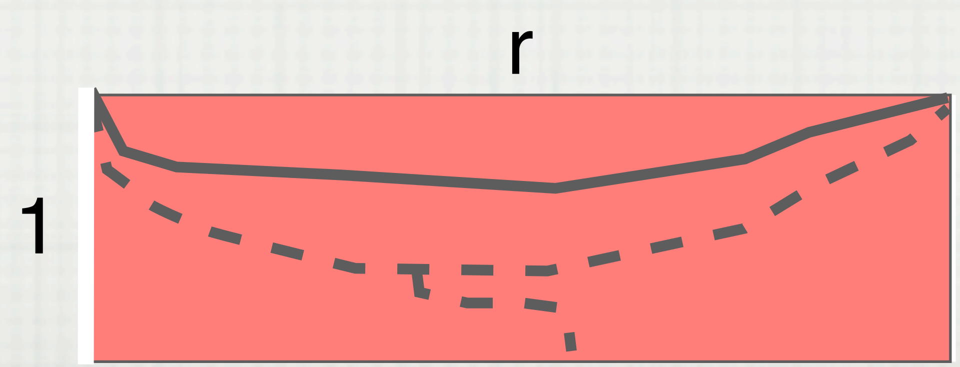
The unusual modular behavior of Π'_{hnv}(r) leads to the definition of a new modular object, the *n^{th}-order* modular form.

Further, Π'_h(r) is completely determined by a simple modular argument that assumes its physical symmetry and generic behavior under T.

These modular properties are surprising, since they occur on a rectangle, which lacks the apparently required symmetry.

NEW CROSSING PROBABILITIES

More recently, Jake Simmons, Bob Ziff, and PK have found three new crossing-type probabilities. We consider the probability density $p_{nb}(\lambda(r))$ of a cluster that connects the upper left and upper right points of the rectangle, with no lower horizontal crossing, but is conditioned to not connect to the bottom.



(Π_{hnv} can be written as a double integral of $p_{nb}(x)$.)

Conformal field theory gives

$$p_{nb}(\lambda(r)) = \frac{(1+\lambda)_2F_1(1,4/3;5/3;\lambda)+2}{4\sqrt{3}\pi(1-\lambda)}$$

(and similar results for two related quantities). We have proven two theorems:

I. $p_{nb}(z)$ is a weakly holomorphic second-order modular form on $\Gamma(2)$ of weight 0 and type $(1,\chi)$.

What does this mean?

Weakly holomorphic: $p_{nb}(z)$ is allowed to diverge exponentially at the cusps. Its leading terms are $(1, q^{-5/6}, q^{2/3})$ at $(\infty, 0, -1)$, respectively.

Second-order of weight 0 and type $(1,\chi)$: Under any elements γ and δ of $\Gamma(2)$,

$$p_{nb}(z)|_{0,1}(\gamma - 1)|_{0,\chi}(\delta - 1) = 0.$$

Here χ is the character of η^4. $\Gamma(2)$ is the group of matrices in $SL_2(\mathbf{Z})$ congruent to I mod(2).

2. To state the next (Hamburger-type) theorem, we first need to define a conformal block (of dimension one). For our purposes, this is just a holomorphic function $P(z)$ with power series expansion

$$P(z) = \Sigma_{n=0}\, a_n\; e^{\pi\, i(n+1)z}$$

with $a_0 \neq 0$.

Set

$$\tilde{P}(z) = P(z) + \frac{1}{4\sqrt{3}}\frac{\lambda'}{\lambda}\left(\frac{\lambda\Pi_h'}{\lambda'}\right)'$$

and suppose

$$\tilde{P}|_4\, g_2 = \tilde{P}$$

along some curve in the upper half-plane, where $g_2 := ST^{-2}S^{-1}$ can be taken as a generator of the group $\Gamma(2)$. Further suppose that $P(-1+i/r)$ and $P(i/r)$ are bounded as $r \to \infty$.

Then

$$P(z) = \frac{(\lambda'(z))^2}{\lambda(z)} p_{nb}(z)$$

Remarks:

A. If we let $z = ir/(1+2ir)$, $r > 0$ be the curve in the condition, then the lhs of the equation is in the physical region, i.e. $z = ir$.

B. The only physical input here is Π'_h. The other two new crossing-type quantities can be characterized with similar theorems. Hence all three can be obtained with only Π'_h as physical input. This suggests some unknown connection between the physical quantities.

There is another interesting result showing the interconnection of these quantities. Define

$$\phi(z) = \frac{C}{2}\frac{(\Pi_{hnv}(\lambda(z))'}{(\Pi_h(\lambda(z))'}$$

(ϕ is in fact a weakly holomorphic second-order modular form of weight 0 and type (1,χ*)). One can show that ϕ depends only on λ:

$$\phi(z) = \frac{1}{2^{8/3}}\lambda(z)^{2/3}{}_2F_1(1/3, 2/3; 5/3; \lambda(z))$$

and further that

$$p_{nb}(z) = \frac{2^{2/3}}{\sqrt{3}\,\pi}\;\frac{1+\lambda(z)}{\lambda(z)^{2/3}\,(1-\lambda(z))^{5/3}}\;\phi(z) + \frac{1}{2\sqrt{3}\,\pi}\;\frac{1}{1-\lambda(z)}$$

i.e., is linear in ϕ with coefficients rational in $\lambda^{1/3}$ and $(1-\lambda)^{1/3}$. The other two crossing-type quantities can be expressed similarly.

SUMMARY

An interesting and surprising connection between physics and modular forms arises in examining crossing and crossing-type formulas in percolation.

REFERENCES

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