Abstract

We consider the iterates of a morphism of projective space. Morton and Silverman conjectured that the number of rational preperiodic points (points with finite orbit) is bounded depending only on the degree of the map, the dimension of the space, and the degree of the number field. The bound in the case of degree two polynomial maps on the projective line over the rational numbers has been extensively considered. Poonen has conjectured that the bound is nine. In this talk we provide strong computational evidence for Poonen's conjecture and examine the case of quadratic extensions. We also discuss how the bound may grow as we allow the dimension of the projective space to increase.

Uniform Boundedness in Arithmetic Dynamics

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(Extended Version)

Outline

1. Dynamical set-up
2. Uniform Boundedness
3. The case of $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ degree 2.
    1. over $\mathbb{Q}$ (previously conjectured)
    2. over $[K : \mathbb{Q}] = 2$ (new)
4. The case of $\phi : \mathbb{P}^N \to \mathbb{P}^N$ degree 2. (new)

References

1. B. Hutz and P. Ingram. Numerical evidence for a conjecture of Poonen. submitted (arXiv:0909.5050).
2. B. Hutz. Rational periodic points for degree two polynomial morphisms on projective space. *Acta Arith.*, forthcoming.

What is Dynamics

Definition

A (discrete) dynamical system is a map $\phi : A \to A$ from a set to itself.

Dynamics is the study of the behavior of points in A under iteration of the map ϕ.

Definition

We write

$$\phi^n = \underbrace{\phi \circ \phi \circ \ldots \circ \phi}_{n}.$$

for the n^{th} iterate of ϕ and we consider the (forward) orbit

$$\mathcal{O}_\phi(a) = \{a, \phi(a), \phi^2(a), \cdots\}.$$

Dynamical Systems on $\mathbb{P}^N$

Definition

A point $P \in \mathbb{P}^N$ is periodic if $\phi^n(P) = P$ for some $n \geq 1$ and preperiodic if $\#\mathcal{O}_\phi(P) < \infty$.

We will denote $\text{Per}(\phi, K)$ and $\text{PrePer}(\phi, K)$ as the K-rational periodic points and preperiodic points for ϕ respectively.

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Definition

A rational map $\phi : \mathbb{P}^N \to \mathbb{P}^N$ is described by homogeneous polynomials of the same degree with no common factor and is a morphism if they also have no common root in $\mathbb{P}^N(K)$. We will denote

$$\phi = [F_0, F_1, \ldots, F_n] : \mathbb{P}^N \to \mathbb{P}^N$$

We will assume $\deg(\phi) = \deg(F_i) \geq 2$.

Northcott: a non-uniform bound

A theorem due to Northcott using a height argument states that

Theorem (Northcott 1950)

Let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism defined over a number field K. Then

$$\mathrm{PrePer}(\phi) \cap \mathbb{P}^N(K) \text{ is finite.}$$

Question

That's nice, but how large can the set of preperiodic points be? If we do not restrict the degree of ϕ we can construct

$$\phi(0) = 1, \phi(1) = 2, \ldots, \phi(n-1) = 0$$

by treating these as linear equations in the coefficients.

Analogy with Elliptic Curves

It is well known that the rational points on an elliptic curve form a group (Mordell-Weil group). In particular, we have a point doubling map

$$\begin{aligned} \phi &: E(K) \to E(K) \\ \phi &: P \mapsto [2]P. \end{aligned}$$

We can think of the torsion points on the elliptic curve as periodic points of ϕ and we can think of the point doubling map as a degree 4 map on $\mathbb{P}^1$ (the x-coordinate).

Merel's theorem states that the number of periodic points for ϕ is bounded uniformly depending only on the degree $[K : \mathbb{Q}]$.

Uniform Bounds

Theorem (Merel)

Let E/K be an elliptic curve defined over a number field K with $[K:\mathbb{Q}] = D$, then there exists a constant $C(D)$ such that

$$\#E(K)_{tors} \leq C.$$

Conjecture (Morton-Silverman, 1994)

Let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree d defined over a number field K with $[K:\mathbb{Q}] = D$. Then there exists a constant $C(d, D, N)$ such that

$$\#PrePer(\phi) \leq C.$$

Quadratic maps on $\mathbb{P}^1$

The case that has been most studied is the case of a single variable polynomial map $f_c(z) = z^2 + c$ (which induces a morphism of $\mathbb{P}^1$).

Definition

Given $g \in \mathrm{PGL}_2(K)$ we define the conjugate of f_c by g as

$$f_c^g = g^{-1} \circ f_c \circ g$$

so that the dynamics is preserved since

$$(f_c^g)^n = g^{-1} \circ f_c^n \circ g.$$

The family f_c for $c \in K$ represents all possible single variable quadratic polynomials over K up to conjugation.

Periodic Points: State of knowledge

We consider the existence of a c such that f_c has a periodic point of primitive period n over $\mathbb{Q}$.

n	Knowledge
1	Yes ... a 1-parameter family
2	Yes ... a 1-parameter family
3	Yes ... a 1-parameter family
4	No - Morton (1998)
5	No - Flynn, Poonen, Schaefer (1997)
6	Conditional No - Stoll (2008)
≥ 7	????

Poonen's Conjecture

Conjecture

For all $n \geq 4$, there is no $c \in \mathbb{Q}$ such that f_c has a rational periodic point of primitive period n over $\mathbb{Q}$.

Theorem (Poonen)

If the conjecture holds, then

$$\#\,\mathrm{PrePer}(f_c, \mathbb{Q}) \leq 9$$

for all $c \in \mathbb{Q}$.

Computational Evidence

Joint with Patrick Ingram:

Definition

The height of a rational number $\frac{p}{q} \in \mathbb{Q}$ is defined as

$$H\left(\frac{p}{q}\right) = \max(|p|, |q|).$$

Theorem (H.,Ingram)

There is no $c \in \mathbb{Q}$ with $H(c) < 10^8$ for which f_c has a periodic point of primitive period ≥ 4 over $\mathbb{Q}$.

Rational maps on $\mathbb{P}^1$

Remark

For a family of degree 2 rational maps on $\mathbb{P}^1$, Manes has a conjecture similar to that of Poonen which states

Conjecture

For all $n \geq 5$, there is no $\phi \in \mathbb{Q}(z)$ such that ϕ has a rational periodic point of primitive period n over $\mathbb{Q}$.

Theorem

Assuming the conjecture

$$\#\operatorname{PrePer}(\phi, \mathbb{Q}) \leq 12.$$

Plan

It is not clear that even checking the conjecture for one choice of c is a finite amount of computation. We will see next how to use information modulo primes to examine this conjecture.

We restrict momentarily to the case R a DVR and K its field of fractions and π a uniformizer and k the reduced field with characteristic p.

Good Reduction

Definition

Given $P = [x_0, \ldots, x_N] \in \mathbb{P}^N(K)$ we choose an α so that αP has each coordinate in R and at least one in R^* and define

$$\tilde{P} = [\widetilde{\alpha x_0}, \ldots, \widetilde{\alpha x_N}] \in \mathbb{P}^N(k)$$

and is independent on the choice of α.

Definition

Given $\phi = [F_0, \ldots, F_N] : \mathbb{P}^N(K) \to \mathbb{P}^N(K)$ we define

$$\tilde{\phi} = [\tilde{F}_0, \ldots, \tilde{F}_N] : \mathbb{P}^N(k) \to \mathbb{P}^N(k).$$

Good Reduction of Maps

Theorem

The following are equivalent

1. $\deg(\phi) = \deg(\tilde{\phi})$
2. *The equations* $\tilde{F}_i = 0$ *have no common solutions in* $\mathbb{P}^N(k)$.
3. $Res(F_0, \ldots, F_n) \in R^*$

Definition

If $\phi : \mathbb{P}^N \to \mathbb{P}^N$ satisfies any condition of the previous theorem, then we say that ϕ has good reduction modulo π.

Good reduction properties

Corollary

If $\phi : \mathbb{P}^N \to \mathbb{P}^N$ has good reduction, then

$$\widetilde{\phi(P)} = \tilde{\phi}(\tilde{P}).$$

and in particular

Corollary

If P is periodic for ϕ of primitive period n, then $\tilde{P}$ is periodic for $\tilde{\phi}$ of primitive period m and $m \mid n$.

A better description of n

Theorem (Morton-Silverman 1994, Zieve)

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map with $\deg(\phi) \geq 2$ defined over K. Assume that ϕ has good reduction and let $P \in \mathbb{P}^1(K)$ be a periodic point for ϕ. Define

$n =$ *primitive period of* P.

$m =$ *primitive period of* P *modulo* π.

$r =$ *the multiplicative order of* $(\phi^m)'(P) \mod \pi$

Then $n = m$, $n = mr$, or $n = mrp^e$ for some $e \geq 1$. Furthermore, if K is characteristic 0 and $v : K^* \to \mathbb{Z}$ is the normalized valuation, then for p odd

$$p^{e-1} \leq \frac{2v(p)}{p-1}.$$

The Algorithm

Applying the previous theorem allows us to produce a finite list of possible periods for f_c for each prime. Intersecting such sets for several primes further restricts the list.

1. Choose a small list of primes to compute the intersected set of possible periods. If the set contains an element ≥ 6, save that c-type.
2. Loop over $c = \frac{a}{b}$ for $H(c) \leq 10^8$.
    1. If b is not a square, then we are already done.
    2. Compute the c-type and compare with list. If not on list, then done. If on list, compute possible periods for more primes and intersect.
    3. If after many primes the conjecture is still not verified, save and leave for manual check (factor($f_c^n(z) = z$) for each remaining possible n).

Denominator of c must be a square

Theorem (Walde,Russo 1994)

Suppose that $z^2 + c$ has a periodic point $\alpha \in \mathbb{A}^1(K)$. Then for each nonarchimedean place v of K with $v(c) < 0$, we have $v(c) = 2v(\alpha)$.

This fact greatly restricts the denominators of c.

Precomputing

Precomputing types eliminates the vast majority of cases:

# primes	# remaining	total number	proportion
1	1	3	0.33333
2	2	12	0.16667
3	5	72	0.06944
4	13	576	0.02257
5	40	6912	0.00579
6	98	96768	0.00101
7	199	1741824	0.00011
8	862	34836480	2.4744×10^{-5}
9	1699	836075520	2.0321×10^{-6}
10	4893	25082265600	1.9508×10^{-7}

Results

Theorem (H.,Ingram)

There is no $c \in \mathbb{Q}$ with $H(c) < 10^8$ for which f_c has a periodic point of primitive period ≥ 4 over $\mathbb{Q}$.

Theorem (H.,Ingram)

For any quadratic field $K/\mathbb{Q}$ with discriminant D satisfying $-4000 \leq D \leq 4000$, and any $c \in K$ with $H(c) \leq 10^3$, $f_c(z) = z^2 + c$ has no K-rational point with primitive period ≥ 7.

Theorem (H.,Ingram)

For any quadratic field $K/\mathbb{Q}$ with discriminant D satisfying $-4000 \leq D \leq 4000$, and any $c \in \mathbb{Q}$ satisfying $H(c) \leq 10^6$, $f_c(z) = z^2 + c$ has no K-rational point with primitive period ≥ 7.

Allowing the dimension to increase

Instead of examining the Morton-Silverman conjecture allowing the degree of the extension to change. Instead, we can allow the dimension of the projective space to change. In other words, consider $\phi : \mathbb{P}^N \to \mathbb{P}^N$ of degree 2 and let $N \to \infty$.

Construction

Theorem (H.)

Let $N \geq 2$. There exist degree two morphisms $\phi : \mathbb{P}^N \to \mathbb{P}^N$ with a $\mathbb{Q}$-rational periodic point of primitive period

$$\begin{cases} 7 = \frac{(N+1)(N+2)}{2} + 1 & \textit{for } N = 2 \\ \frac{(N+1)(N+2)}{2} + \left\lfloor \frac{N-1}{2} \right\rfloor & \textit{for } N \geq 3, \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Increasing Bound

1. Construct an explicit family of polynomial maps.
2. Verify with resultants that each family contains at least one morphism.

Since the maps are polynomials ($F_N = x_N^2$) we can “combine” morphisms to get another morphism.

Theorem (H.)

For any $k \geq 1$, there is a constant $c(k)$ such that for all N large enough, there exists a degree two morphism of $\mathbb{P}^N$ with a $\mathbb{Q}$-rational periodic point with primitive period larger than $c(k)N^k$.

Construction Method

Notice that by taking a generic quadratic polynomial

$$f(x) = ax^2 + bx + c,$$

we could try to force $x = 0$ to be a periodic point with primitive period 3 by solving the system of equations

$$\{f(0) \neq 0, f^2(0) \neq 0, f^3(0) = 0\}.$$

For example, we could choose $c = 1$ and then $a = -b + 1$, and then we are left with a linear equation in b for $f^3(0) = 0$.

$$f^3(0) = -2b + 5.$$

Construction for $\mathbb{P}^2$

For more general maps of projective space, we can perform a similar construction to find a map $\phi : \mathbb{P}^N \to \mathbb{P}^N$ where $[0, \ldots, 0, 1]$ is a periodic point of primitive period $\binom{N+2}{2}$, the number of coefficients of a quadratic form in $N+1$ variables.
However, we can do better. We demonstrate for $\mathbb{P}^2$
Choose coefficients so that

$$[0,0,1] \xrightarrow{\phi} [1,0,1] \xrightarrow{\phi} [0,1,1] \xrightarrow{\phi} [1,1,1].$$

Continued

Then we choose

$$\phi([1,1,1]) = [0, C_1, 1]$$

and then

$$\phi([0, C_1, 1])) = [0, C_2, 1].$$

So we have three points chosen of the form $[0, x_1, 1]$ whose images are given by

$$[0,0,1] \xrightarrow{\phi} [1,0,1]$$

$$[0,1,1] \xrightarrow{\phi} [1,1,1]$$

$$[0, C_1, 1] \xrightarrow{\phi} [0, C_2, 1].$$

Continued

Since $x_i(\phi[0, x_1, 1])$ is a quadratic polynomial in x_1 for $0 \leq i \leq 1$, the image $\phi([0, C_2, 1])$ is determined by these three known points and is of the form

$$\phi([0, C_2, 1]) = [k_0, k_1, 1]$$

for some constants k_0 and k_1. So we have that each $x_i(\phi([k_0, k_1, 1]))$ is linear in the remaining coefficient and we choose it so that

$$\phi([k_0, k_1, 1]) = [0, 0, 1].$$

This is a primitive 7-periodic point.

In general

After the initial set-up, every other choice of coefficients allows the primitive period to increase by 2 instead of 1.
We can find:

N	trivial bound	Construction bound	example period
2	6	7	9
3	10	11	24
4	15	16	72

Recall Growth of the Bound

Theorem (H.)

For any $k \geq 1$, there is a constant $c(k)$ such that for all N large enough, there exists a degree two morphism of $\mathbb{P}^N$ with a $\mathbb{Q}$-rational periodic point with primitive period larger than $c(k)N^k$.

Proof

Fix k a positive integer and set $s = k/2$. Let $M = \lfloor N/s \rfloor$. Then

$$\frac{(M+1)(M+2)}{2} > \frac{(N/s)(N/s)}{2} = \frac{N^2}{2s^2}$$

and for every prime $p \leq \frac{N^2}{2s^2}$ there is a point with primitive period p for some polynomial morphism of $\mathbb{P}^M$.

Proof Continued

Fix $\epsilon > 0$ and choose N large enough that the interval

$$((1-\epsilon)N^2/2s^2, N^2/2s^2)$$

has at least s primes $p_1, \ldots, p_s$. We combine these points and associated morphisms to get a point $P \in \mathbb{P}^{sM} = \mathbb{P}^N$, which has primitive period

$$p_1 \cdots p_s \geq \frac{(1-\epsilon)^s}{2^s s^{2s}} N^{2s}$$

for a polynomial morphism of $\mathbb{P}^N$.

Example

The point $[0,0,0,0,1] \in \mathbb{P}^4$ is a periodic point of primitive period 72 for the morphism by combining periodic points of primitive period 8 and 9 in $\mathbb{P}^2$.

$$\begin{aligned}
&\phi([x_0, x_1, x_2, x_3, x_4]) = \\
&\quad [-\frac{38}{45}x_0^2 + (2x_1 - \frac{7}{45}x_4)x_0 - \frac{1}{2}x_1^2 - \frac{1}{2}x_4x_1 + x_4^2, \\
&\quad -\frac{67}{90}x_0^2 + (2x_1 + \frac{157}{90}x_4)x_0 - x_4x_1, \\
&\quad (-x_3 - x_4)x_2 - \frac{13}{30}x_3^2 + \frac{13}{30}x_4x_3 + x_4^2, \\
&\quad -\frac{1}{2}x_2^2 + (-x_3 + \frac{3}{2}x_4)x_2 - \frac{1}{3}x_3^2 + \frac{4}{3}x_4x_3, \\
&\quad x_4^2].
\end{aligned}$$