Ramification in iterated towers for rational functions

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Basic Question

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Example. $\varphi(x) = x^2 - 3$ does **not** have this property.

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- Gives rise to the cyclotomic $\mathbf{Z}_2$-extension of $\mathbf{Q}$
- Key observation: *ramification is finite*

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- Observation: The K_n defined by the $\phi^{(n)}$ are also finitely-ramified

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Question. Given this setup, describe the ramification and Galois properties of the iterated towers generated by φ.

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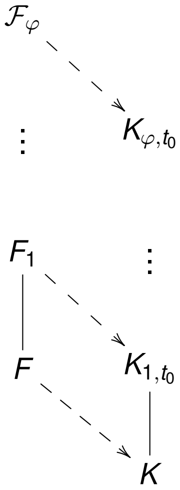

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Definition. ϕ is *postcritically finite* if the forward orbit of the critical points of ϕ is finite.

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- **Problem.** Compute $\mathrm{disc}(g_n(x) - t h_n(x))$.

Theorem

Let $\varphi(x) = g(x)/h(x) \in K(x)$ be postcritically finite with coprime polynomials $g(x), h(x)$. Then for each $t_0 \in K$, there exists a *finite* set S_{t_0} of primes of K such that for all $n \geq 1$, if $\mathfrak{p}$ is a prime of K not in S_{t_0}, then $v_{\mathfrak{p}}(\mathrm{disc}(g_n(x) - th_n(x))) = 0$.

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Corollary

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Generalizes the theorem of Aitken, Hajir, and Maire.

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First compute

$$\mathrm{disc}(g(x) - th(x)) =$$
$$\frac{(-1)^{\binom{m}{2}+\epsilon\delta-mq+m\epsilon}\ell^{\epsilon+m-q-2}D^m}{\ell(h)^{m-\delta}\,\mathrm{Res}(g(x),h(x))}\prod_{r\in\mathcal{R}_\varphi}(g(r)-th(r))^{m_r}.$$

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$$\prod_{r\in\mathcal{R}_{\phi(n)}}(g_n(r)-th_n(r))^{m_r} =$$
$$\pm\,(\ell(h_n)\ell(g_n)(\delta-\epsilon)^n)^{\delta^n}\operatorname{Res}(g_n,h_n)\ell(g_n)\operatorname{disc}(g_n)\prod_{\beta\in\mathcal{B}_{\phi(n)}}(1-t/\beta)^{M_\beta}.$$

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Example. The Rikuna polynomials

$$p_n(x) - tq_n(x) = \frac{\zeta^{-1}(x-\zeta)^n - \zeta(x-\zeta^{-1})^n}{\zeta^{-1}-\zeta} - t\frac{(x-\zeta)^n - (x-\zeta^{-1})^n}{\zeta^{-1}-\zeta}$$

have important arithmetic applications

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Set $\varphi = p(x)/q(x)$ and study iterates $\varphi^{(n)}(x)$.